111B Section Week 9

Overview: Work on the following problems one at a time, either by yourself or in small-groups. After a sufficient amount of time has passed, we will discuss the solutions as a class. Attending section counts toward your participation grade.

1. Let R be a commutative ring and $I \subseteq R$ an ideal. The *radical* of I is the set is the set $\operatorname{rad} I = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$. An ideal is called *radical* if $\operatorname{rad} I = I$.
   (a) Prove that $\operatorname{rad} I$ is an ideal containing I and that $(\operatorname{rad} I)/I = \mathfrak{N}(R/I)$.
   (b) Prove that every prime ideal is radical.
2. Let R be a commutative ring with 1. The *spectrum* $\operatorname{Spec} R$ is the set of all prime ideals of R. For any ideal $I \subseteq R$, define
$$V(I) = \{P \in \operatorname{Spec} R : I \subseteq P\}.$$
   (a) For any ideals I, J of R, show that $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.
   (b) If $\{I_\alpha\}$ is any collection of ideals of R, show that $V(\sum_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$.
3. Let R and S be a commutative ring with 1 and let $\varphi : R \to S$ be a unital ring homomorphism.
   (a) Show that there is a function $\varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R$ defined by the rule $\varphi^*(P) = \varphi^{-1}(P)$ for any $P \in \operatorname{Spec} R$.
   (b) For any ideal I of R, show that $(\varphi^*)^{-1}(V(I)) = V(\varphi(I)S)$ where $\varphi(I)S \subseteq S$ is the ideal generated by $\varphi(I)$.